Some Linear Multi-step Methods for the Initial Value Problems $y'' = f(x, y, y')$ by Perturbed Collocation

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Abstract-Two linear multi-step schemes for the numerical solutions of initial values problems of the type $y'' = f(x, y, y')$ by perturbed collocation using Legendre and Chebyshev polynomials as our approximating functions in tau methods of solution were developed. The schemes were found to perform very well when compared with existing known schemes and were also found to be stable.

Index Terms- Analytical solution, Canonical polynomials, Chebyshev-tau method, Legendre-tau method, Multi-step methods, Ordinary differential equations, Perturbed collocation

1 INTRODUCTION

Many of the real-life problems arising from the fields such as Engineering, Natural Sciences, Environmental and Social Sciences, Economics and other humanities can be modeled by either linear or non-linear (or quasi-linear) ordinary differential equations ''[1], [3], [4], [6]''. Therefore, this calls for the use of efficient and effective numerical methods, particularly when such models are transformed to appropriate mathematical problems.

In this work, we shall derive some linear multi-step methods (LMM) for initial value problems (IVP) of second order o.d.e. by perturbed collocation.

Consider the general form of this type of equation in the form

$$y''(x) = f(x, y, y') \qquad a \le x \le b$$
$$y(0) = y_0, y'(0) = y'_0 \tag{1}$$

A special case of this problem is

$$y'' = f(x, y) \qquad a \le x \le b$$
$$y(0) = y_0, y'(0) = y'_0 \tag{2}$$

[7] looked into equation (2) and came up with two schemes (3) and (4) for the solution of this problem using Legendre and Chebyshev polynomials respectively.

$$y_{i+2} - 2y_{i+1} + y_i = \frac{h^2}{6}\{f_{i+2} + 4f_{i+1} + f_i\} \tag{3}$$

and

$$y_{i+2} - 2y_{i+1} + y_i = \frac{h^2}{4}\{f_{i+2} + 2f_{i+1} + f_i\} (4)$$

These schemes were found to perform very well but they were found to be incapable of handling problems of the type (1). Their problem was based on the fact that their method could not deal with the case when the RHS of 1 involves the first derivate. The fact that real-life problems are not limited to equations of the type (2) according to ''[2], [5]'' we decided to find a way of overcoming this shortcoming so that we could have an algorithm that will handle these two forms of equations to a good degree of accuracy.

2 DERIVATION OF THE SCHEMES

In equation (1), we assume that

i. the equation has unique solution
ii $h = x_{i+1} - x_i$ is constant
iii $y(x_0) = y_0, y(x_{i+j}) = y_{i+j}, y'(x_{i+j}) = y'_{i+j}, f(x_{i+j}, y_{i+j}) = f_{j+j}$

We then seek a solution of the form

$$y_N(x) = \sum_{j=0}^{N} a_j Q_j(x) \tag{5}$$

where N is the degree of the polynomial $y_N(x)$ and $Q_j(x)$, j = 0, 1, 2, … are canonical polynomials [8] generated by the operator

$$\Im = \frac{d^2}{dx^2} + \frac{d}{dx} + 1 \tag{6}$$

Define

$$\Im Q_j(x) = x^j$$

then, we can easily show that the following recurrence relation holds

$$Q_j(x) = x^j - j(j-1)Q_{j-2}(x) - j(Q_{j-1}(x); \quad j = 0, 1, 2, ...(7)$$

Suppose $N = 2$ (for second order differential equations), then (5) can be written as

$$y_2(x) = a_0 + a_1(x-1) + a_2(x^2 - 2x), \qquad x_{i-1} \le x \le x_{i+1} \quad (8)$$

We define the residual function

$$R_N(x) \text{ as}: R_N(x) = \tau_{N-1}P_{N-1}(x) + \tau_N P_N(x) \quad N > 1 \quad (9)$$

which can be approximated by appropriate orthogonal polynomial. Substituting (8) in (1) we have

$$a_1 + 2a_2(x-1) + 2a_2 = f(x, y, y') + R_2(x) \quad (10)$$

2.1 Legendre-tau Approximant

We substitute (9) in (10) for N =2 to obtain

$$a_1 + 2a_2 x = f(x, y) + \tau_1 P_1(x) + \tau_2 P_2(x) \quad (11)$$

where P_N is Legendre polynomial of degree N.

Collocating (11) at $x = x_{i-1}, x = x_i$ and $x = x_{i+1}$ and interpolating (80) at $x = x_{i-1}$ and $x = x_i$ we obtain the following perturbed collocation equations

$$\begin{aligned} a_1 + 2a_2 x_{i-1} &= f_{i-1}\tau_1 P_1(x_{i-1}) + \tau_2 P_2(x_{i-1}) \\ a_1 + 2a_2 x_i &= f_{i1} + \tau_1 P_1(x_{i_i}) + \tau_2 P_2(x_i) \\ a_1 + 2a_2 x_{i+1} &= f_{i+1} + \tau_1 P_1(x_{i+1}) + \tau_2 P_2(x_{i+1}) \\ y_2(x_{i-1}) &= a_0 + a_1(x_{i-1} - 1) + a_2(x_{i-1}^2 - 2x_{i-1}) \\ y_2(x_i) &= a_0 + a_1(x_1) + a_2(x_1^2 - 2x_i) \end{aligned} \quad (12)$$

The polynomials P_1 and P_2 defined in [-1, 1] are then transformed into the interval [x_{i-1}, x_{i+1}]. Equations (12) are then solved for a_i , i = 0, 1, 2 and τ_j , j = 1, 2 so that (8) becomes

$$y_2(x_{i+1}) = \frac{4}{(2+h)} y_i - \frac{(2-h)}{(2+h)} y_{i-1} + \frac{h^2}{3(2+h)}[f_{i+1} + 4f_i + f_{i-1}] \quad (13)$$

$h \neq -2$. Replacing i by i + 1 in (13) we have

$$y_{i+2} - \frac{4}{(2+h)} y_{i+1} + \frac{(2-h)}{(2+h)} y_1 = \frac{h^2}{3(2+h)}[f_{i+2} + 4f_{i+1} + f_i] \quad (14)$$

Equation (14) is an implicit discrete formula used in solving (1). The error constant of the method is $\frac{-h^4}{6(2+h)}$ and it is of order 2. Its region of stability covers [-12, 0].

2.2 Chebyshev-tau Approximant

We define $R_N(x)$ in Chebyshev term as

$$R_N(x) = \tau_{N-1}, \quad T_{N-1}(x) + \tau_N T_N(x); \quad N \ge 1 \quad (15)$$

If N = 2, we can write (8) as

$$a_1 + 2a_2 x = f(x, y, y') + \tau_1 T_1(x) + \tau_2 T_2(x)$$

(16)

Again, collocating (16) at $x = x_{i-1}, x = x_i$ and $x = x_{i+1}$ we obtain

$$\begin{aligned} a_1 + 2a_2 x_{i-1} &= f_{i-1} + \tau_1 T_1(x_{i-1}) + \tau_2 T_2(x_{i-1}) \\ a_1 + 2a_2 x_i &= f_i + \tau_1 T_1(x_i) + \tau_2 T_2(x_i) \\ a_1 + 2a_2 x_{i+1} &= f_{i+1} + \tau_1 T_1(x_{i+1i}) + \tau_2 T_2(x_{i+1}) \\ y_2(x_{i-1}) &= a_0 + a_1(x_{i-1} - 1) + a_2(x_{i-1}^2 - 2x_{i-1}) \\ y_2(x_i) &= a_0 + a_1(x_{i-1}) + a_2(x_i^2 - 2x_i) \end{aligned} \quad (17)$$

The polynomials T_i defined in [-1, 1] are first transformed into the interval [x_{i-1}, x_{i+1}] then substituted into (15). The resulting equations are then solved for the unknowns so that (5) can be written as

$$y_2(x_{i+1}) = \frac{4}{(2+h)} y_i - \left(\frac{2-h}{2+h}\right) y_{i-1} + \frac{h^2}{2(2+h)}[f_{i+1} + 2f_i + f_i] \quad (18)$$

On replacing i by i + 1 we have

$$y_{i+2} - \frac{4}{(2+h)} y_{i+1} + \left(\frac{2-h}{2+h}\right) y_i = \frac{h^2}{2(2+h)}[f_{i+2} + 2f_{i+1} + f_i] \quad (19)$$

Equation (19) is again an implicit discrete formula used in solving (1). The error constant of the method is $\frac{-h^4}{6(2+h)}$ and it is also of order 2. The stability region of the method is [-∞, 0].

3 NUMERICAL EXPERIMENT

Two problems are used in our experiment.The first problem is of the type (2) while the second is of the type (1).

Problem 1

$y'' = y; \quad y(0) = 1, y'(0) = 1, x \in [0,1]$, h = 0.1

Analytical solution is $y = e^x$

Problem 2

$y'' + y' - 2y = \sin x, \quad y(0) = 4, y'(0) = 4, x \in [0,1], h = 0.1$

Analytical solution is $y = \frac{25}{6}e^x + \frac{1}{12}e^{-2x} - \frac{1}{4}\cos x$

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TABLE 1

SOLUTION OF PROBLEM (1) USING GENERAL CASES (14) AND (19) AND THE SPECIAL CASES (3) AND (4)

X	General cases Legendre (equation 2.6)	 Chebyshev (equation 3.6)	Special cases Legendre (equation 1.3)	 Chebyshev (equation 1.4)
0.0	0.000000000×10^{0}	0.000000000×10^{0}	0.000000000×10^{0}	0.000000000×10^{0}
0.1	$1.304000000x10^{-6}$	$1.304000000x10^{-6}$	$1.304000000x10^{-6}$	$1.304000000x10^{-6}$
0.2	$1.054078312x10^{-2}$	$1.053654149x10^{-2}$	$2.316000000x10^{-6}$	$6.769332500x10^{-6}$
0.3	$1.163983975x10^{-2}$	$1.163063555x10^{-2}$	$1.249756167x10^{-5}$	$2.216197250x10^{-5}$
0.4	$2.506330703x10^{-2}$	$2.476434219x10^{-2}$	$5.887433587x10^{-4}$	$8.922730200x10^{-4}$
0.5	$3.596509010x10^{-3}$	$4.514135785x10^{-3}$	$1.771619300x10^{-3}$	$2.677146135x10^{-3}$
0.6	0.090041670×10^{0}	$8.825350582x10^{-2}$	$3.596509010x10^{-3}$	$4.514135785x10^{-3}$
0.7	$1.418157777x10^{-1}$	$1.383179677x10^{-1}$	$7.711124430x10^{-3}$	$8.822160573x10^{-3}$
0.8	$2.045757502x10^{-1}$	$1.971799740x10^{-1}$	$1.515289005x10^{-2}$	$2.002504479x10^{-2}$
0.9	$2.76941288x10^{-1}$	$2.6221680972x10^{-1}$	$3.087834911x10^{-2}$	$4.265570100x10^{-2}$
1.0	$3.578826883x10^{-1}$	$3.310201873x10^{-1}$	$5.741798914x10^{-2}$	$8.143455555x10^{-2}$

TABLE 2

ERRORS IN PROBLEM (1) USING GENERAL CASES (14) AND (19) AND THE SPECIAL CASES (3) AND(4)

X	General cases Legendre (equation 2.6)	 Chebyshev (equation 3.6)	Special cases Legendre (equation 1.3)	 Chebyshev (equation 1.4)
0.0	0.000000000×10^{0}	0.000000000×10^{0}	0.000000000×10^{0}	0.000000000×10^{0}
0.1	$1.304000000x10^{-6}$	$1.304000000x10^{-6}$	$1.304000000x10^{-6}$	$1.304000000x10^{-6}$
0.2	$1.054078312x10^{-2}$	$1.053654149x10^{-2}$	$2.316000000x10^{-6}$	$6.769332500x10^{-6}$
0.3	$1.163983975x10^{-2}$	$1.163063555x10^{-2}$	$1.249756167x10^{-5}$	$2.216197250x10^{-5}$
0.4	$2.506330703x10^{-2}$	$2.476434219x10^{-2}$	$5.887433587x10^{-4}$	$8.922730200x10^{-4}$
0.5	$3.596509010x10^{-3}$	$4.514135785x10^{-3}$	$1.771619300x10^{-3}$	$2.677146135x10^{-3}$
0.6	0.090041670×10^{0}	$8.825350582x10^{-2}$	$3.596509010x10^{-3}$	$4.514135785x10^{-3}$
0.7	$1.418157777x10^{-1}$	$1.383179677x10^{-1}$	$7.711124430x10^{-3}$	$8.822160573x10^{-3}$
0.8	$2.045757502x10^{-1}$	$1.971799740x10^{-1}$	$1.515289005x10^{-2}$	$2.002504479x10^{-2}$
0.9	$2.76941288x10^{-1}$	$2.6221680972x10^{-1}$	$3.087834911x10^{-2}$	$4.265570100x10^{-2}$
1.0	$3.578826883x10^{-1}$	$3.310201873x10^{-1}$	$5.741798914x10^{-2}$	$8.143455555x10^{-2}$

TABLE 3

SOLUTION OF PROBLEM (2) USING GENERAL CASES (14) AND (19) AND THE SPECIAL CASES (3) AND (4)

X	General cases Legendre (equation 2.6)	 Chebyshev (equation 3.6)	Special cases Legendre (equation 1.3)	 Chebyshev (equation 1.4)
0.0	$1.904656430x10^{-1}$	$1.904656430x10^{-1}$	0.000000000×10^{0}	0.00000000×10^{0}
0.1	$3.894125485x10^{-2}$	$3.344156277x10^{-2}$	$7.315453222x10^{-3}$	$7.315453222x10^{-3}$
0.2	$1.033313860x10^{-1}$	$8.218303910x10^{-2}$	$1.025823425x10^{-2}$	$1.033286126x10^{-2}$
0.3	$1.875686188x10^{-6}$	$1.363872455x10^{-1}$	$9.919233170x10^{-2}$	$9.989583045x10^{-2}$
0.4	$2.856462596x10^{-1}$	$1.859934026x10^{-1}$	$1.600338640x10^{-1}$	$1.784071329x10^{-1}$
0.5	$3.931299927x10^{-1}$	$2.229072769x10^{-1}$	$3.109254963x10^{-1}$	$3.110114269x10^{-1}$
0.6	$5.067881753x10^{-1}$	$2.404480508x10^{-1}$	$4.883858890x10^{-1}$	$5.234922800x10^{-1}$
0.7	$6.243019529x10^{-1}$	$2.329074939x10^{-1}$	$7.179635114x10^{-1}$	$7.180717350x10^{-1}$

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0.8	7.440382163x10^{-1}	1.951967200x10^{-1}	1.008817549×10^{0}	1.008366028×10^{0}
0.9	8.648718558x10^{-1}	1.225596309x10^{-1}	1.369320215×10^{0}	1.369457881×10^{0}
1.0	1.904656430x10^{-1}	1.90465643x10^{-1}	1.8125685339×10^{0}	1.812514091×10^{0}

TABLE 4

ERRORS IN PROBLEM (2) USING GENERAL CASES (14) AND (19) AND THE SPECIAL CASES (3) AND (4)

X	General cases		Special cases	
	Legendre (equation 2.6)	Chebyshev (equation 3.6)	Legendre (equation 1.3)	Chebyshev (equation 1.4)
0.0	1.904656430x10^{-1}	1.904656430x10^{-1}	0.000000000×10^{0}	0.00000000×10^{0}
0.1	3.894125485x10^{-2}	3.344156277x10^{-2}	7.315453222x10^{-3}	7.315453222x10^{-3}
0.2	1.033313860x10^{-1}	8.218303910x10^{-2}	1.025823425x10^{-2}	1.033286126x10^{-2}
0.3	1.875686188x10^{-6}	1.363872455x10^{-1}	9.919233170x10^{-2}	9.989583045x10^{-2}
0.4	2.856462596x10^{-1}	1.859934026x10^{-1}	1.600338640x10^{-1}	1.784071329x10^{-1}
0.5	3.931299927x10^{-1}	2.229072769x10^{-1}	3.109254963x10^{-1}	3.110114269x10^{-1}
0.6	5.067881753x10^{-1}	2.404480508x10^{-1}	4.883858890x10^{-1}	5.234922800x10^{-1}
0.7	6.243019529x10^{-1}	2.329074939x10^{-1}	7.179635114x10^{-1}	7.180717350x10^{-1}
0.8	7.440382163x10^{-1}	1.951967200x10^{-1}	1.008817549×10^{0}	1.008366028×10^{0}
0.9	8.648718558x10^{-1}	1.225596309x10^{-1}	1.369320215×10^{0}	1.369457881×10^{0}
1.0	1.904656430x10^{-1}	1.90465643x10^{-1}	1.8125685339×10^{0}	1.812514091×10^{0}

4 DISCUSSION OF RESULTS

The errors defined as $e_n = \left| y(x) - y_n \right|$ are illustrated in tables 2 and 4 for problems 1 and 2 respectively. The special schemes (3) and (4) perform better than the general schemes (13) and (14) when applied to solve special second order ordinary differential equation (2). On the contrary, they proved weak to handle the general cases of ordinary differential equation (1) as illustrated in table 1. However, the general schemes (13) and (14) attempted to solve special case (2) of ordinary differential equation averagely and proved much better in handling general case (1) compared with the special schemes (3) and (4). Nevertheless, the special and the general schemes cover same region of stability and same order but differs in error constant.

5 Conclusion

The results of this work as illustrated by problems (1) and (2) show that an algorithm for the solution of initial value problem using the combination of (3) and (14) or (4) and (19) is possible. If the problem is of the type (1), either (3) or (4) is called on to solve the problem but if the problem is of the type (2), either (14) or (19) is called on. The combination of these schemes will act as a veritable tool for the numerical solution of initial value problems.

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